Resolving the Universe with Multifractals

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Abstract

We present a new method of quantifying structure in any discreet point set. The method is based on a multifractal approach and includes correction for all relevant boundary and selection effects. The power of this technique lies in its connection to higher order moments, in that it not only probes clustering on different scales but also different densities.

Multifractal Formalism

The universe has already been shown to be well described using a multifractal framework [Jones *et al.* ApJ Lett., **332**, 1988]. In this analysis we will adopt the procedure laid out in Hentschel & Procaccia [Phys. Rev. A, **27**, 1266 1983] to determine the Rényi (Generalised) dimensions of a point set embedded in a three-dimensional Euclidean space. The probability of a galaxy, j, being within a sphere of radius r centred on galaxy i, is,

$$p_i(r) = \frac{n_i(r)}{N} = \frac{1}{N}\sum_{j=1}^{N}\Theta(|r_i - r_j| - r) \tag{1}$$

Here $n_i(r)$ is the number of galaxies within radius r, N is the total number of galaxies and Θ is the Heaviside step function. Eq.1 can then be related to the conditional partition function [Grassberger & Procaccia. Phys. Rev. A, **28**, 1983.],

$$Z(q,r) = \frac{1}{M}\sum_{i=1}^{M}[p_i(r)]^{q-1} \propto r^{\tau(q)} \tag{2}$$

In this case M is the number of counting spheres and q defines the generalised dimension we are investigating. $\tau(q)$ is the scaling exponent, which is then related to the infinite set of Rényi dimensions through,

$$D_q = \frac{\tau(q)}{q-1}, \quad q \neq 1 \tag{3}$$

Clearly the special case of $q = 1$, the information dimension, cannot be determined using the above expression but can be found approximately in the limit $q \to 1$. This is an important dimension to calculate as it gives equal weighting to voids and clusters. Voids are enhanced for $q < 1$ and clusters are enhanced for $q > 1$, so $q = 1$ is, in some sense, the most unbiased dimension in the set.

Boundary Corrections

Various methods have been proposed to correct for the 'missing' galaxies outside the boundary of a survey. Here we will briefly introduce our own method.

Volume Correction

In considering a the counting sphere, of radius r, which extends beyond the geometrical boundary of the survey. The number of galaxies counted in the sphere of radius is, therefore, depleted. To correct for this problem we could either add galaxies to the missing region or we could somehow modify the volume. We can recast Eq.1 as,

$$p_i(r) = \frac{V_i(r)\rho^\star(r)}{N} = \frac{V_i(r)}{V_i^\star(r)}\cdot\frac{n_i^\star(r)}{N} \tag{4}$$

Here $V^\star$ is the reduced volume. On its own this method would assign to the missing region the same average density as the rest of the sphere. This would be wrong if density varies with distance. To overcome this problem we assume *only* that the density does not vary with θ or ϕ i.e. the universe is isotropic and hence Eq.4 will hold for fixed r. So to apply our method to a galaxy survey we must count in spherical shells, correcting in each shell and then integrating over radius. The principle behind this method is illustrated in figure 1. The shells are individually corrected and summed according to,

$$p_i(r) = \sum_{r=0}^{r}\alpha_i(r)\frac{n_i^\star(r)}{N}$$

Here $\alpha_i(r) \equiv \frac{V}{V^\star}$; this is the enhancement factor of the i^{th} shell at radius r and has value ≥ 1.

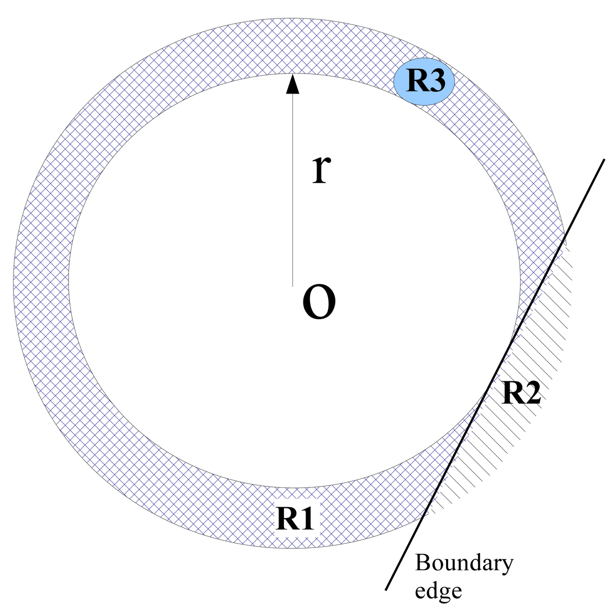
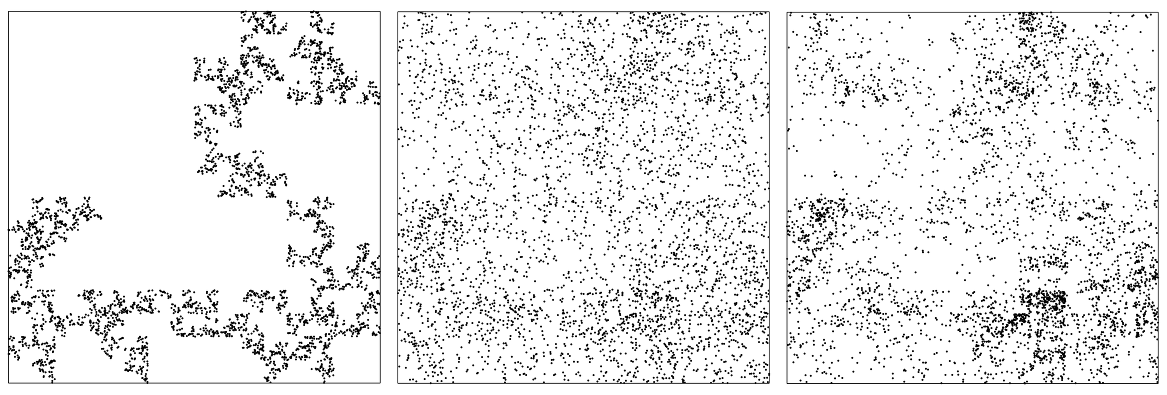
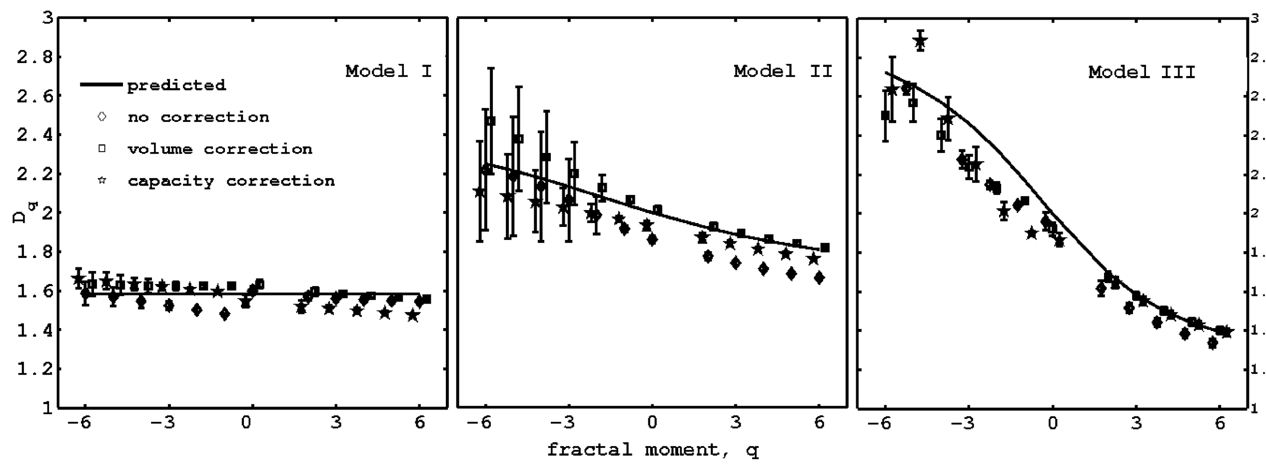


Figure 1: A counting sphere centred on a galaxy with radius, r. Region R2 is outside the survey and R3 is a masked region.

Toy Model

Below we present examples of a toy fractal model based on the Multiplicative Cascade.





As can be seen in the lower panels, the Renyi dimensions estimated using our volume correction method are in excellent agreement with the predicted curves. It can be shown [Martínez *et. al.* ApJ 357 1990] that as $L \to \infty$,

$$D_q = \frac{\log_2(f_1^q + f_2^q + f_3^q + f_4^q)}{1-q}; \qquad f_i = \frac{p_i}{\sum_i p_i}. \tag{5}$$

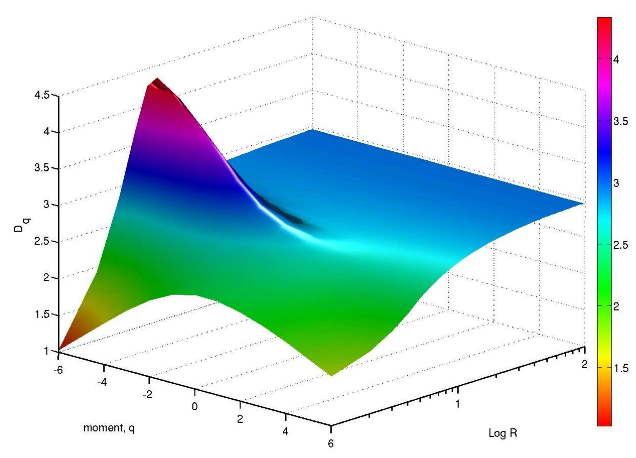
Where the p_i's are input parameters. The above fractal images have the following parameters:

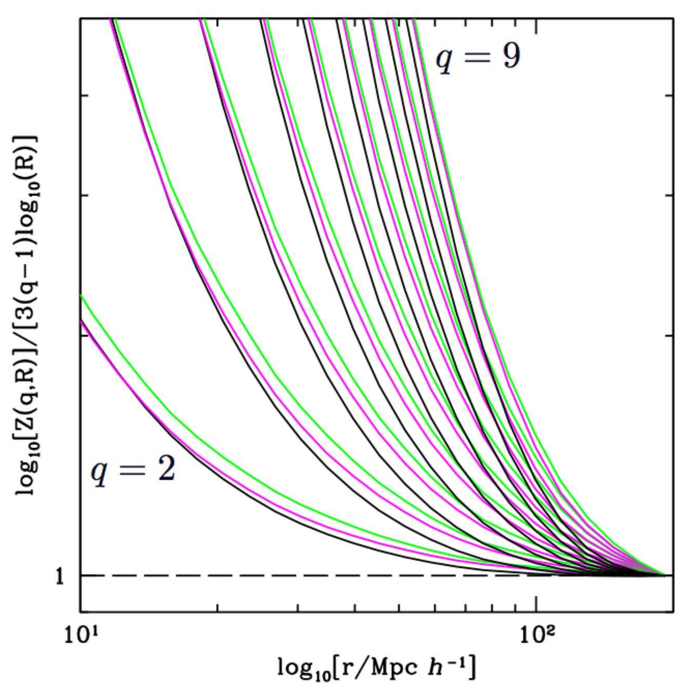
Model	p_1	p_2	p_3	p_4
I	1	1	1	0
II	1	0.75	0.75	0.5
III	1	0.5	0.5	0.25

Thus we feel justified in applying this analysis to galaxy redshift surveys despite their complicated geometry. To begin with however we will analyse an ideal case; a Halo Catalog.

D_{qR} Surface

We now extend the idea of the D_q curve to vary with scale. This gives us a D_{qR} *Surface*. Here we have applied the multifractal analysis to N-body simulation and halos catalogues. The upper right plot shows the $D_{q,R}$ surface for the halo positions in an ideal and complete *(384 Mpc)*$^3h^{-3}$ box with a flat geometry and cosmological parameters $p = (\Omega_m, \Omega_b, n, h, \sigma_8) = (0.3, 0.04, 1, 0.7, 0.9)$ [Warren *et al* ApJ **646**, 881, 2006]. Such a surface has been calculate using dark matter only. Overall, it is a very smooth surface, which tends towards homogeneity on large scales. A clear peak, at low q, and dip, at high q corresponds to multifractality for R $\approx$ 10Mpc . The lower plot shows the "normalised" conditional partition sum for $2 < q < 9$. The dark matter (dark line) and halos homogeneity contributions [$\sim R^{-3(q-1)}$] have been factored out. Halos were extracted from the simulation using a FOF with a linking length of 0.2. With different HOD parameters we reproduced different galaxy and group populations: SDSS groups (green) and SDSS galaxies (magenta).





Summary and Conclusions

We have:

- Developed a new method for optimally correcting masked regions and boundaries in redshift surveys.
- Introduced the Mutifractal Formalism and extended it to produce the D_{qR} surface.
- Showed that we can recover the underlying Renyi dimensions of a multifractal distribution while using our boundary correction.
- Applied the methodology to a Halo catalogs.

In future work we plan to analyse SDSS data, and in particular the LRG population, using their clustering properties – characterised by the D_{qR} surface – as a cosmological probe.