On the Lagrangian theory of cosmological density perturbations

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Outline

- Cosmological model
- Scalar perturbations
- Hydrodynamical approach
- Field approach
- Isocurvature perturbations
- Conclusions

Cosmological model

Background Friedmann-Robertson-Walker metrics,
Spatially flat Universe:

$$ds^2 = dt^2 - a^2 dx_\alpha dx^\alpha = a^2(d\eta^2 - dx_\alpha dx^\alpha),$$

$$d\eta = dt/a.$$

Friedmann equations (which are Einstein equations for the FRW metrics):

$$H^2 = \frac{8\pi G}{3}\varepsilon,$$

$$\gamma \equiv -\frac{\dot{H}}{H^2} = \frac{3}{2}\left(1 + \frac{p}{\varepsilon}\right).$$

Scalar and tensor perturbations

Generally, the metrics perturbations can be split into irreducible representations which correspond to scalar, vector and tensor perturbations.

Scalar perturbations describe density perturbations.
Vector perturbations correspond to perturbations of vortical velocity .
Tensor perturbations correspond to gravitational waves.

Here we focus on scalar perturbations.

Scalar perturbations of the metrics and energy-momentum tensor.

$$h_{ij} = \begin{pmatrix} 2D & aC_{,\alpha} \\ aC_{,\beta} & 2a^2(A\delta_{\alpha\beta} + B_{,\alpha\beta}) \end{pmatrix}.$$

$$\delta T_0^0 = \delta\varepsilon,$$

$$\delta T_\alpha^0 = (\varepsilon + p)v_{,\alpha},$$

$$-\delta T_\beta^\alpha = \delta p \delta_{\alpha\beta} + (\varepsilon + p)\sigma_{\alpha\beta},$$

$$\sigma_{\alpha\beta} = \frac{1}{2a^2H^2}(E_{,\alpha\beta} - \triangle E\delta_{\alpha\beta}), \quad \sigma^\beta_{\alpha,\beta} = 0,$$

■, ■, ■, ■, ■, ■, ■, ■ — 8 scalar potentials.

Gauge transformations

Splitting in “background” and “perturbation” is not unique. With coordinate transformations, we obtain different background and different perturbation. Hence, unphysical perturbations may arise.

$$Q = Q^{(0)} + \delta Q$$

In “a bit” different reference frame:

$$Q = \widetilde{Q}^{(0)} + \widetilde{\delta Q}$$

Gauge-invariant variables

Almost all of the metrics and material potentials are not gauge-invariant, but one can construct gauge-invariant variables from them. One of the variables is q-scalar:

$$q = A + Hv.$$

(V.N. Lukash, 1980)

q-scalar is constructed from the gravitational part A which is prominent at large scales and a hydrodynamical part (second term) which is prominent at small scales.

Inverse transformations from q-scalar to the initial potentials

The material potentials and the metrics potentials are not independent. They are linked through perturbed Einstein equations:

[redacted]

β

The inverse transformations are as follows:

$$v = \frac{q - A}{H}, \qquad \delta p_c \equiv \delta p - \dot{p} v = \frac{\varepsilon + p}{H} \dot{q},$$

$$\delta \varepsilon_c \equiv \delta \varepsilon - \dot{\varepsilon} v = \frac{\triangle \Phi}{4\pi G a^2}, \qquad a\dot{B} - C = \frac{A - \Phi}{aH},$$

$$D = \gamma q - \frac{d}{dt}\left(\frac{A}{H}\right), \qquad \Phi = \frac{H}{a} \int a \gamma q dt.$$

Thus, there are 10 unknowns for 6 equations. We then set E=0 (isotropic pressure), and a gauge-transformation contains two arbitrary scalar functions:

[illegible]

Now there is the only unknown left.

One extra equation can be obtained in two ways.
The first way (hydrodynamical approach) is
to write a relation between comoving gauge-invariant perturbations of pressure and energy density.

$$\delta p_c = \beta^2(t)\delta\varepsilon_c$$

The second way (field approach) is
to write a quite arbitrary Lagrangian for the phi-field:

$$\mathcal{L} = \mathcal{L}(\varphi, w).$$

Field approach

One immediately has the energy-momentum tensor.

$$T_k^i = (\varepsilon + p)u^i u_k - p\delta_k^i,$$

$$\varepsilon = nw - \mathcal{L}, \quad p = \mathcal{L}, \quad n \equiv \frac{\partial \mathcal{L}}{\partial w},$$

$$u_i = \frac{\varphi_{,i}}{w},$$

Field approach

$$\frac{\delta\varepsilon_c}{\delta p_c} = \frac{n_{,w} w \delta w - n_{,w} \dot{w} \delta\varphi}{n \delta w - n \dot{w} w^{-1} \delta\varphi} = \frac{n_{,w} w}{n} =$$

$$= \frac{w}{n} \frac{\partial^2 \mathcal{L}}{\partial w^2} \equiv c_s^{-2}(\varphi, w) = c_s^{-2}(t).$$

Thus, the two approaches are equivalent to first order.

Dynamical equation for q

With both approaches, we obtain the following equation for evolution of the q field:

$$\ddot{q} + \left(3H + 2\frac{\dot{\alpha}}{\alpha}\right)\dot{q} - \left(\frac{\beta}{a}\right)^2 \triangle q = 0.$$

$$\alpha^2 = \frac{\gamma}{4\pi G\beta^2}$$

In the field approach one should substitute beta for c_s:

$$c_s^{-2} = \frac{w}{n}\frac{\partial^2 \mathcal{L}}{\partial w^2}, \quad n = \frac{\partial \mathcal{L}}{\partial w}.$$

Action and Lagrangian of perturbations

The perturbation action is quite simple:

$$S[q] = \frac{1}{2}\int \alpha^2 \left(\dot{q}^2 - \left(\frac{\beta}{a}\right)^2 q_{,i}q^{,i} \right) a^3 dt d^3x$$

That is, q is a test “massless” scalar field.

Isocurvature perturbations

With several media, perturbations that do not perturb
curvature
are also possible.
These are isocurvature (isothermic, entropy) perturbations.

$$L = p(w_1, w_2) = p_1(w_1) + p_2(w_2), \quad w_1^2 = g^{ik}\varphi_{1,i}\varphi_{1,k}, \quad w_2^2 = g^{ik}\varphi_{2,i}\varphi_{2,k}.$$

$$T_{ik} = (n_1 w_1 + n_2 w_2) u_i u_k - p g_{ik}.$$

$$u_i = \frac{n_1 w_1 u_{1i} + n_2 w_2 u_{2i}}{n_1 w_1 + n_2 w_2}.$$

The Lagrangian for isocurvature perturbations

$$L(q,r) = \frac{1}{2}\alpha_q^2 a^3 \left(\dot{q}^2 - \frac{1}{a^2\overline{\beta^{-2}}} q_{,\alpha} q^{,\alpha}\right) + \frac{1}{2}\frac{\alpha_r^2 H^2}{a^3 n_1 w_1 n_2 w_2}\left(\dot{r}^2 - \frac{1}{a^2\overline{\beta^2}\beta_1^{-2}\beta_2^{-2}} r_{,\alpha} r^{,\alpha}\right) +$$

$$+\frac{(\beta_1^{-2} - \beta_2^{-2})}{2H}\dot{r}\dot{q},$$

$$q = \frac{n_1 w_1 q_1 + n_2 w_2 q_2}{\varepsilon + p}, \quad r = \frac{a^3 n_1 w_1 n_2 w_2}{H(\varepsilon + p)}(q_1 - q_2).$$

$$\beta_1^{-2} = \frac{n_{1,w_1} w_1}{n_1}, \beta_2^{-2} = \frac{n_{2,w_2} w_2}{n_2}.$$

Equations of motion

$$\begin{cases} \Box_{c_q}\overline{q} = \dfrac{(\alpha_q a)''}{\alpha_q a}\overline{q} - \dfrac{1}{2\alpha_q a}\left(\dfrac{(\beta_1^{-2}-\beta_2^{-2})}{aH}\left(\dfrac{\overline{r}}{\Omega}\right)'\right)', \\ \\ \Box_{c_r}\overline{r} = \dfrac{\Omega''}{\Omega}\overline{r} - \dfrac{1}{2\Omega}\left(\dfrac{(\beta_1^{-2}-\beta_2^{-2})}{aH}\left(\dfrac{\overline{q}}{\alpha_q a}\right)'\right)', \end{cases}$$

$$\overline{q} = \alpha_q a q, \quad \overline{r} = \Omega r, \quad \Omega = \frac{\alpha_r H}{a^2\sqrt{n_1 w_1 n_2 w_2}}.$$

$$\alpha_q^2 = \gamma\overline{\beta^{-2}}, \; \alpha_r^2 = \gamma\overline{\beta^2}\beta_1^{-2}\beta_2^{-2}$$

«Normal» modes

$$L(q_1, q_2) = \frac{1}{2}\frac{\gamma\beta_1^{-2}a^3 n_1 w_1}{\varepsilon + p}\left(\dot{q}_1^2 - \frac{\beta_1^2}{a^2}q_{1,\alpha}q_1^{;\alpha}\right) + \frac{1}{2}\frac{\gamma\beta_2^{-2}a^3 n_2 w_2}{\varepsilon + p}\left(\dot{q}_2^2 - \frac{\beta_2^2}{a^2}q_{2,\alpha}q_2^{;\alpha}\right) +$$

$$+L_{int}(q_1, q_2).$$

$$L_{int} = \frac{1}{2}(S - T)(\dot{q}_1 q_2 - \dot{q}_2 q_1) + (q_1 - q_2)(U q_1 - W q_2).$$

Conclusions

- Hydrodynamical and field approaches are equivalent to first order of the cosmological scalar perturbations theory.
- The Lagrangian for adiabatic and isocurvature modes has been built. It appears that the isocurvature mode also has a speed of sound.


Thank you!